

Optimal L^∞ error estimates for finite element Galerkin methods for nonlinear evolution equations

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Abstract Finite element Galerkin solutions for three classes of nonlinear evolution equations are considered. The existence, uniqueness and convergence of the fully discrete Crank-Nicolson scheme are discussed. At last a linearized Galerkin approximation is presented, which is also second order accurate in time fully discrete scheme.

Keywords Galerkin method · Crank-Nicolson scheme · Existence · Uniqueness · Convergence · Extrapolation · Linearization

Mathematics Subject Classification (2000) 65M06 · 65M15 · 65N30

1 Introduction

We consider three classes of nonlinear non stationary equations: The Kuramoto-Tsuzuki equation describes the behavior of many two-component systems in a neighborhood of the bifurcation point [10]. Reaction-diffusion type equations have been applied in the study of broad class of nonlinear processes, including a well-known synergetic model [2, 11] and the nonlinear heat equation. The problem of constructing and validating difference schemes for these classes of problems has been taken in detail up in [8, 9], see also [5, 6, 22, 24] and [17–19]. Let $\Omega =]0, 1[$ and $Q = \Omega \times]0, T[$. We consider the initial-boundary value problem for the nonlinear evolution problem:

$$u_t = A \frac{\partial^2 u}{\partial x^2} + f(u), \quad (x, t) \in Q, \quad (1.1.1)$$

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$$u(0, t) = u(1, t) = 0, \quad 0 < t \leq T, \quad (1.1.2)$$

$$u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \quad (1.1.3)$$

where $u = (u_1, u_2, \dots, u_r)$ is a complex vector-function, A is a complex diagonal matrix, f is a complex vector-function and u_0 is a given complex valued function.

Here we have the following cases:

- (1) If $\text{Im}(A) = 0$ we have the reaction diffusion type equation.
- (2) If $\text{Re}(A) > 0$ and $\text{Im}(A) \neq 0$ we have the Kuramoto-Tsuzuki equation.
- (3) If $\text{Re}(A) > 0$ and $\text{Im}(A) = 0$ we have a heat equation.

Since the matrix A is diagonal, there is no essential difference between the study of system (1.1.1) and the study of one equation. Therefore, we shall consider the single equation:

$$u_t = a \frac{\partial^2 u}{\partial x^2} + f(u), \quad (x, t) \in Q, \quad (1.2.1)$$

$$u(0, t) = u(1, t) = 0, \quad 0 < t \leq T, \quad (1.2.2)$$

$$u(x, 0) = u_0(x), \quad x \in \bar{\Omega}. \quad (1.2.3)$$

We assume that:

$$(i) \quad a \in \mathbb{C}, \text{ and } \text{Re}(a) = \alpha > 0, \quad (1.2.4)$$

$$(ii) \quad f(u) = b|u|^2 u, \quad (1.2.5)$$

$$(iii) \quad b \in \mathbb{C}, \text{ and } \text{Re}(b) = \beta < 0. \quad (1.2.6)$$

The condition (1.2.4) means the positivity of the heat conduction coefficient.

An outline of the remaining part of the paper is as follows: In Sect. 2, after explaining notation, the numerical scheme is described in detail. The existence of the fully discrete finite element is shown in Sect. 3. After some priori estimates presented in Sect. 4, one proves the uniqueness of the approximate solution in Sect. 5. Optimal rates of convergence (second-order in time) are derived in Sect. 6. At last section, a linearized Galerkin method is presented which is also convergent in the L^∞ -norm.

2 Crank-Nicolson Galerkin method

Throughout the paper, we use D to denote $\frac{\partial}{\partial x}$. The norms of $L^2(\Omega)$, $L^\infty(\Omega)$ and $H^s(\Omega)$ are denoted by $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_s$. The semi-norm $\|D^s v\|$ is denoted by $|v|_s$, $(v, w) = \int_\Omega v \bar{w} dx$ denotes the inner product of $L^2(\Omega)$.

Let r and l be integers with $r \geq 2$ and $0 \leq l \leq r - 2$, and consider a family of partitions $0 = x_0 < x_1 < x_2 < \dots < x_l = 1$ of $[0, 1]$ into subintervals $J_i = (x_{i-1}, x_i)$, set

$$h = \max_{1 \leq i \leq l} (x_i - x_{i-1}),$$

and S_h be the piecewise polynomial spline space:

$$S_h = \{\chi \in C^l[0, 1], \chi|_{J_i} \in P_{r-1}(J_i), i = 1, \dots, I, \chi(0) = \chi(1) = 0\},$$

where $P_{r-1}(J_i)$ denotes the set of polynomials on J_i of degree less than or equal to $r - 1$.

We first consider a discretization in time based on Crank-Nicolson Galerkin method. For any given positive integer N , let $k = \frac{T}{N}$ denote the size of the time discretization and $t^n = nk$, $k = 0, 1, \dots, N$. For a smooth function ϕ on $[0, T]$, let $\phi^n = \phi(t^n)$, $\partial_t \phi^n = \frac{1}{k}(\phi^n - \phi^{n-1})$ and $\phi^{n-\frac{1}{2}} = \frac{1}{2}(\phi^n + \phi^{n-1})$.

Our discrete time Galerkin approximation U^n of $u(t^n)$ is now defined as a solution of

$$(\partial_t U^n, \chi) = -a(DU^{n-\frac{1}{2}}, D\chi) + b(\varphi(U^{n-\frac{1}{2}}), \chi), \quad \forall \chi \in S_h, \quad (2.1.1)$$

with

$$U^0 = u_{0h}, \quad (2.1.2)$$

where $u_{0h} \in S_h$ is an appropriate approximation to u_0 and $\varphi(z) = |z|^2 z$.

3 Existence

We shall next prove the existence of a sequence $\{U^n\}_{n=0}^N$ satisfying (2.1). For this, we shall use the following variant of the well-known fixed point theorem of Brouwer [3, 4].

Lemma 1 *Let H be a finite dimensional space with inner product $(\cdot, \cdot)_H$, and norm $\|\cdot\|_H$. Let the map $g : H \rightarrow H$ be continuous. Suppose there exists $\alpha > 0$ such that $\operatorname{Re}(g(Z), Z)_H \geq 0$ for all Z with $\|Z\|_H = \alpha$. Then there exists $Z^* \in H$, such that $g(Z^*) = 0$ and $\|Z^*\| \leq \alpha$.*

Theorem 1 *The approximate solution U^n of (2.1.1)–(2.1.2) exists.*

Proof In order to prove the Theorem by the mathematical induction. Obviously U^0 exists. Moreover assume $\{U^j\}_{j=0}^{n-1}$ exists.

For $Z \in S_h$, define $g : S_h \rightarrow S_h$ by

$$(g(Z), \chi) = (Z - U^{n-1}, \chi) + \frac{k}{2}[a(DZ, D\chi) - b(|Z|^2 Z, \chi)], \quad \forall \chi \in S_h. \quad (3.1)$$

Such a map exists by the Riesz representation theorem and g is obviously continuous. Taking $\chi = Z$, in (3.1) we obtain

$$(g(Z), Z) = \|Z\|^2 - (U^{n-1}, Z) + \frac{k}{2}(a|Z|_1^2 - b\|Z\|_{L^4}^4).$$

Therefore,

$$\operatorname{Re}(g(Z), Z) = \|Z\|^2 - \operatorname{Re}(U^{n-1}, Z) + \frac{k}{2}(\alpha|Z|_1^2 - \beta\|Z\|_{L^4}^4).$$

Using the assumptions (1.2.4) and (1.2.6), we find

$$\begin{aligned}\operatorname{Re}(g(Z), Z) &\geq \|Z\|^2 - \|U^{n-1}\| \|Z\| \\ &= \|Z\|(\|Z\| - \|U^{n-1}\|).\end{aligned}$$

With $\|Z\| = \|U^{n-1}\| + 1$, we have $\operatorname{Re}(g(Z), Z) > 0$. Now an application of Lemma 1 yields an existence of a $Z^* \in S_h$ such that $g(Z^*) = 0$. It is easily seen that $U^n = 2Z^* - U^{n-1}$ satisfies (2.1). This completes the proof of existence. \square

4 Stability

In this section, we will discuss the stability of numerical solution to (2.1).

Lemma 2 Suppose hypothesis (1.2.4), (1.2.5) and (1.2.6) are satisfied. Then for the solution of problem (2.1) the following estimate holds:

$$\|U^n\| \leq \|U^0\|, \quad 1 \leq n \leq N.$$

Proof Choosing $\chi = U^{n-\frac{1}{2}}$ in (2.1.1), we have

$$(\partial_t U^n, U^{n-\frac{1}{2}}) = -a|U^{n-\frac{1}{2}}|_1^2 + b\|U^{n-\frac{1}{2}}\|_{L^4}^4.$$

Taking the real part, we find from (1.2.4) and (1.2.6)

$$\frac{1}{2k}(\|U^n\|^2 - \|U^{n-1}\|^2) \leq 0.$$

Therefore,

$$\|U^n\|^2 \leq \|U^0\|^2.$$

This completes the proof. \square

Next we will use the Nirenberg inequality [1].

Lemma 3 For $\frac{j}{m} \leq a \leq 1$, $\frac{1}{p} = \frac{j}{n'} + a(\frac{1}{r} - \frac{m}{n'}) + (1-a)\frac{1}{q}$, there holds:

$$\|D^j v\|_{L^p(\Omega)} \leq C[\|D^m v\|_{L^r(\Omega)}^a \|v\|_{L^q(\Omega)}^{1-a} + \|v\|_{L^q(\Omega)}],$$

where Ω is a bounded domain in $\mathbb{R}^{n'}$.

Lemma 4 For any $x \geq 0$, $y \geq 0$ and $p \geq 1$, there holds:

$$(x + y)^p \leq 2^{p-1}(x^p + y^p).$$

Theorem 2 Let $\{U^n\}_{n=0}^N$ be the solution of (2.1). Assume that the initial approximation $u_{0h} \in H_0^1(0, 1)$. Then we have the following estimate for small k and $n = 1, 2, \dots, N$, $\|U^n\|_\infty \leq c_0$, where $c_0 = c_0(u_{0h}, T, |b|, \alpha)$.

Proof Multiplying (2.1.1) by $\frac{1}{a}$ and taking $\chi = \partial_t U^n$, we get

$$\frac{1}{a} \|\partial_t U^n\|^2 = -(DU^{n-\frac{1}{2}}, D(\partial_t U^n)) + \frac{b}{a} (\varphi(U^{n-\frac{1}{2}}), \partial_t U^n).$$

Taking the real part, we find

$$\begin{aligned} \frac{\alpha}{|a|^2} \|\partial_t U^n\|^2 &= -\frac{1}{2k} (|U^n|_1^2 - |U^{n-1}|_1^2) + \operatorname{Re} \left[\frac{b}{a} (\varphi(U^{n-\frac{1}{2}}), \partial_t U^n) \right] \\ &\leq -\frac{1}{2k} (|U^n|_1^2 - |U^{n-1}|_1^2) + \frac{|b|}{|a|} \|U^{n-\frac{1}{2}}\|_{L^6}^3 \|\partial_t U^n\| \\ &\leq -\frac{1}{2k} (|U^n|_1^2 - |U^{n-1}|_1^2) + \frac{|b|^2}{4\alpha} \|U^{n-\frac{1}{2}}\|_{L^6}^6 + \frac{\alpha}{|a|^2} \|\partial_t U^n\|^2. \end{aligned}$$

This yields,

$$\frac{1}{k} (|U^n|_1^2 - |U^{n-1}|_1^2) \leq \frac{|b|^2}{2\alpha} \|U^{n-\frac{1}{2}}\|_{L^6}^6. \quad (4.1)$$

Applying Lemma 3 with $p = 6$, we find

$$\|U^{n-\frac{1}{2}}\|_{L^6}^6 \leq \{C[\|U^{n-\frac{1}{2}}\|^{\frac{2}{3}} |U^{n-\frac{1}{2}}|_1^{\frac{1}{3}} + \|U^{n-\frac{1}{2}}\|]\}^6.$$

By Lemma 4, we have

$$\|U^{n-\frac{1}{2}}\|_{L^6}^6 \leq 2^5 C^6 [\|U^{n-\frac{1}{2}}\|^4 |U^{n-\frac{1}{2}}|_1^2 + \|U^{n-\frac{1}{2}}\|^6]. \quad (4.2)$$

Using (4.1), (4.2) and Lemma 2, we obtain

$$\begin{aligned} \frac{1}{k} (|U^n|_1^2 - |U^{n-1}|_1^2) &\leq C |U^{n-\frac{1}{2}}|_1^2 + C' \\ &\leq \frac{1}{2} C (|U^n|_1^2 + |U^{n-1}|_1^2) + C'. \end{aligned}$$

For small k , we find by Gronwall's inequality

$$|U^n|_1^2 \leq C(T) |U^0|_1^2 + C'(T). \quad (4.3)$$

Using Lemma 3 with $p = \infty$ and Lemma 2 and (4.3), we obtain

$$\|U^n\|_\infty \leq c_0. \quad (4.4)$$

This completes the proof. \square

5 Uniqueness

We shall show global uniqueness of the approximations U^1, U^2, \dots, U^N satisfying (2.1). We assume that the initial approximation u_{0h} is sufficiently regular.

Theorem 3 *The approximate solution U^n of (2.1.1)–(2.1.2) is unique.*

Proof Let $V^n \in S_h$ be another solution of (2.1) with $V^0 = u_{0h}$. Then V^n satisfies that

$$(\partial_t V^n, \chi) = -a(DV^{n-\frac{1}{2}}, D\chi) + b(\varphi(V^{n-\frac{1}{2}}), \chi), \quad \forall \chi \in S_h. \quad (5.1)$$

Let $E^i = U^i - V^i$, with $E^0 = 0$. Then using (2.1.1) and (5.1), we have for $\chi \in S_h$

$$(\partial_t E^n, \chi) = -a(DE^{n-\frac{1}{2}}, D\chi) + b(\varphi(U^{n-\frac{1}{2}}) - \varphi(V^{n-\frac{1}{2}}), \chi). \quad (5.2)$$

The proof of uniqueness is with the induction method. Now, supposing $E^{n-1} = 0$ and choosing $\chi = E^{n-\frac{1}{2}}$ in (5.2), we obtain

$$\frac{1}{2k}(\|E^n\|^2 - \|E^{n-1}\|^2) = -a|E^{n-\frac{1}{2}}|_1^2 + b(\varphi(U^{n-\frac{1}{2}}) - \varphi(V^{n-\frac{1}{2}}), E^{n-\frac{1}{2}}).$$

Taking the real part, we have

$$\frac{1}{2k}(\|E^n\|^2 - \|E^{n-1}\|^2) = -\alpha|E^{n-\frac{1}{2}}|_1^2 + \operatorname{Re}[b(\varphi(U^{n-\frac{1}{2}}) - \varphi(V^{n-\frac{1}{2}}), E^{n-\frac{1}{2}})].$$

From (1.2.4), we get

$$\frac{1}{2k}(\|E^n\|^2 - \|E^{n-1}\|^2) \leq |b| |(\varphi(U^{n-\frac{1}{2}}) - \varphi(V^{n-\frac{1}{2}}), E^{n-\frac{1}{2}})|. \quad (5.3)$$

Using the inequality

$$||z_1|^2 z_1 - |z_2|^2 z_2| \leq (|z_1| + |z_2|)^2 |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}. \quad (5.4)$$

We have by Theorem 2,

$$|(\varphi(U^{n-\frac{1}{2}}) - \varphi(V^{n-\frac{1}{2}}), E^{n-\frac{1}{2}})| \leq 4c_0^2 \|E^{n-\frac{1}{2}}\|^2.$$

Substituting this inequality into (5.3), we obtain

$$\|E^n\|^2 - \|E^{n-1}\|^2 \leq 8kc_0^2 |b| \|E^{n-\frac{1}{2}}\|^2,$$

from which, for k sufficiently small, we have

$$\|E^n\| \leq \lambda \|E^{n-1}\|,$$

where

$$\lambda = \left\{ \frac{1 + 4kc_0^2 |b|}{1 - 4kc_0^2 |b|} \right\}^{\frac{1}{2}}.$$

We see that $E^n = 0$. This completes the proof of uniqueness of solutions for (2.1). \square

6 Rate of convergence estimates

In order to discuss the convergence of the discretizations, it is useful to introduce the auxiliary projection $P_h : H_0^1(\Omega) \rightarrow S_h$ defined by: for $v \in H_0^1(\Omega)$

$$(D(P_h v - v), D\chi) = 0, \quad \forall \chi \in S_h. \quad (6.1)$$

Then, as is well-known, $P_h v$ enjoys the following properties [16]:

Lemma 5 *With P_h defined by (6.1), we have*

$$\|P_h v - v\| \leq Ch^r \|v\|_r, \quad (6.2.1)$$

$$\|D(P_h v - v)\| \leq Ch^{r-1} \|v\|_r, \quad (6.2.2)$$

$$\|P_h v - v\|_\infty \leq Ch^r \|v\|_{W_\infty^r(\Omega)}. \quad (6.2.3)$$

We shall want to estimate the error in the fully discrete problem (2.1), henceforth the solution of (1.2) and u_0 are sufficiently regular, similar results were obtained in [12–15]. Our approach is based on the error decomposition with $u^n = u(t^n)$:

$$U^n - u^n = (U^n - P_h u^n) + (P_h u^n - u^n) = \theta^n + \rho^n,$$

(see [20, 23, 24] for linear parabolic equations). Denote

$$c'_0 = \max_{0 \leq x \leq 1, 0 \leq t \leq T} |u(x, t)|.$$

Theorem 4 *Let U^n and u be the solutions of (2.1) and (1.2) respectively. Suppose that the solution u is sufficiently regular. If the initial data satisfy*

$$P_h u^0 = u_{0h}, \quad (6.3)$$

then, for h, k sufficiently small,

$$\|U^n - u(t^n)\| \leq C(h^r + k^2), \quad (6.4)$$

where $C = C(u, T, \alpha, |a|, |b|, c_0, c'_0)$.

Proof Since the estimates of ρ^n are obtained from (6.2.1), it is enough to estimate θ^n . Using the elliptic projection (6.1) and equations (1.2.1) and (2.1.1), we obtain the following error equation in θ^n :

$$\begin{aligned} (\partial_t \theta^n, \chi) = & -a(D\theta^{n-\frac{1}{2}}, D\chi) + b(\varphi(U^{n-\frac{1}{2}}) - \varphi(u^{n-\frac{1}{2}}), \chi) \\ & - (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \chi) - a\left(D\left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\right), D\chi\right). \end{aligned} \quad (6.5)$$

Setting $\chi = \theta^{n-\frac{1}{2}}$, we have

$$\begin{aligned} (\partial_t \theta^n, \theta^{n-\frac{1}{2}}) &= -a|\theta^{n-\frac{1}{2}}|_1^2 + b(\varphi(U^{n-\frac{1}{2}}) - \varphi(u^{n-\frac{1}{2}}), \theta^{n-\frac{1}{2}}) \\ &\quad - a \left(D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right), D\theta^{n-\frac{1}{2}} \right) \\ &\quad - (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \theta^{n-\frac{1}{2}}). \end{aligned} \quad (6.6)$$

Using Theorem 2 and (5.4), we obtain

$$\begin{aligned} |\varphi(U^{n-\frac{1}{2}}) - \varphi(u^{n-\frac{1}{2}})| &\leq (|U^{n-\frac{1}{2}}| + |u^{n-\frac{1}{2}}|)^2 |U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}| \\ &\leq (c_0 + c'_0)^2 |U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}|, \end{aligned}$$

Taking the real part in (6.6) and using the last inequality, we find

$$\begin{aligned} \frac{1}{2k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) &\leq -\alpha|\theta^{n-\frac{1}{2}}|_1^2 + \frac{1}{2}|b|^2(c_0 + c'_0)^4\|\theta^{n-\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2}\|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\|^2 + \frac{1}{2}\|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2}\|\theta^{n-\frac{1}{2}}\|^2 + \frac{|a|^2}{2\alpha} \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|^2 \\ &\quad + \frac{1}{2}\alpha|\theta^{n-\frac{1}{2}}|_1^2. \end{aligned}$$

This yields by (1.2.4)

$$\frac{1}{2k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq C[\|\theta^{n-\frac{1}{2}}\|^2 + (\sigma^n)^2 + (\delta^n)^2 + (\xi^n)^2], \quad (6.7)$$

where

$$\begin{aligned} \sigma^n &= \|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\|, \\ \delta^n &= \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\|, \\ \xi^n &= \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|. \end{aligned}$$

We have the following estimates:

$$\begin{aligned} \sigma^n &\leq \|\theta^{n-\frac{1}{2}}\| + \left\| P_h \left(\frac{u^n + u^{n-1}}{2} \right) - u^{n-\frac{1}{2}} \right\| \\ &\leq \|\theta^{n-\frac{1}{2}}\| + \frac{1}{2}(\|\rho^n\| + \|\rho^{n-1}\|) + \left\| \frac{1}{2}(u^n + u^{n-1}) - u^{n-\frac{1}{2}} \right\| \\ &\leq \|\theta^{n-\frac{1}{2}}\| + \frac{1}{2}(\|\rho^n\| + \|\rho^{n-1}\|) + Ck^{\frac{3}{2}} \left(\int_{t_{n-1}}^{t_n} \|u_{tt}(s)\|^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

where C is a generic constant independent of step sizes h and k .

$$\begin{aligned}\delta^n &\leq \|\partial_t P_h u^n - \partial_t u^n\| + \|\partial_t u^n - u_t^{n-\frac{1}{2}}\| \\ &= \frac{1}{k} \|\rho^n - \rho^{n-1}\| + \left\| \frac{1}{k} (u^n - u^{n-1}) - u_t^{n-\frac{1}{2}} \right\| \\ &\leq k^{-\frac{1}{2}} \left(\int_{t_{n-1}}^{t_n} \|\rho_t(s)\|^2 ds \right)^{\frac{1}{2}} + Ck^{\frac{3}{2}} \left(\int_{t_{n-1}}^{t_n} \|u_{ttt}(s)\|^2 ds \right)^{\frac{1}{2}}.\end{aligned}$$

In the same way as above, we have

$$\xi^n = \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\| \leq Ck^{\frac{3}{2}} \left(\int_{t_{n-1}}^{t_n} \|Du_{tt}(s)\|^2 ds \right)^{\frac{1}{2}}.$$

Using (6.7) with the estimates of σ^n , δ^n and ξ^n we obtain

$$\begin{aligned}&\frac{1}{2k} (\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \\ &\leq C \left[\|\theta^{n-\frac{1}{2}}\|^2 + \|\rho^n\|^2 + \|\rho^{n-1}\|^2 + \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t(s)\|^2 ds \right. \\ &\quad \left. + k^3 \int_{t_{n-1}}^{t_n} (\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|Du_{tt}(s)\|^2) ds \right].\end{aligned}$$

Hence

$$\|\theta^n\|^2 - \|\theta^{n-1}\|^2 \leq Ck(\|\theta^{n-\frac{1}{2}}\|^2 + R_n).$$

Here the latter equality defines R_n , so that

$$(1 - Ck)\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + CkR_n,$$

and for k sufficiently small,

$$\|\theta^n\|^2 \leq \left(\frac{1 + Ck}{1 - Ck} \right) \|\theta^{n-1}\|^2 + CkR_n.$$

After repeated application, this yields

$$\begin{aligned}\|\theta^n\|^2 &\leq \left(\frac{1 + Ck}{1 - Ck} \right)^n \|\theta^0\|^2 + Ck \sum_{j=1}^n \left(\frac{1 + Ck}{1 - Ck} \right)^{n-j} R_j \\ &\leq C\|\theta^0\|^2 + Ck \sum_{j=1}^n R_j.\end{aligned}$$

Using (6.3), the above inequality becomes

$$\|\theta^n\|^2 \leq C \left[k \sum_{j=1}^n \|\rho^j\|^2 + \int_0^T \|\rho_t(s)\|^2 ds \right]$$

$$+ k^4 \int_0^T (\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|Du_{tt}(s)\|^2) ds \Big],$$

and by (6.2.1)

$$\|\theta^n\| \leq C(u, T)(h^r + k^2), \quad 1 \leq n \leq N, \quad (6.8)$$

which completes the proof. \square

The main result of this paper is given in the following Theorem.

Theorem 5 *Let the conditions of Theorem 4 be satisfied. Then, for h, k sufficiently small*

$$\|U^n - u(t^n)\|_\infty \leq C(h^r + k^2),$$

where $C = C(u, T, \Omega, \alpha, |a|, |b|, c_0, c'_0)$.

Proof Choosing $\chi = \partial_t \theta^n$ in (6.5), multiplying by $\frac{1}{a}$ and taking the real part, we have

$$\begin{aligned} \frac{\alpha}{|a|^2} \|\partial_t \theta^n\|^2 &= -\frac{1}{2} \partial_t |\theta^n|_1^2 + \operatorname{Re} \left[\frac{b}{a} (\varphi(U^{n-\frac{1}{2}}) - \varphi(u^{n-\frac{1}{2}}), \partial_t \theta^n) \right] \\ &\quad - \operatorname{Re} \left[\frac{1}{a} (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \partial_t \theta^n) \right] \\ &\quad - \operatorname{Re} \left[\left(D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right), D(\partial_t \theta^n) \right) \right] \\ &\leq -\frac{1}{2} \partial_t |\theta^n|_1^2 + \frac{|b|}{|a|} (c_0 + c'_0)^2 \|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\| \|\partial_t \theta^n\| \\ &\quad + \frac{1}{|a|} \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\| \|\partial_t \theta^n\| \\ &\quad + \frac{1}{4} \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|^2 + |\partial_t \theta^n|_1^2. \end{aligned}$$

It follows from the Friedrichs inequality

$$\begin{aligned} \left(\frac{1}{C(\Omega)} + \frac{\alpha}{|a|^2} \right) \|\partial_t \theta^n\|^2 &\leq -\frac{1}{2} \partial_t |\theta^n|_1^2 + \frac{|b|}{|a|} (c_0 + c'_0)^2 \|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\| \|\partial_t \theta^n\| \\ &\quad + \frac{1}{|a|} \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\| \|\partial_t \theta^n\| \\ &\quad + \frac{1}{4} \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|^2. \end{aligned}$$

Therefore

$$\begin{aligned}
 \frac{|a|^2 + \alpha C(\Omega)}{|a|^2 C(\Omega)} \|\partial_t \theta^n\|^2 &\leq -\frac{1}{2k} (|\theta^n|_1^2 - |\theta^{n-1}|_1^2) + \frac{1}{2} \frac{|a|^2 + \alpha C(\Omega)}{|a|^2 C(\Omega)} \|\partial_t \theta^n\|^2 \\
 &\quad + \frac{1}{2} \frac{(c_0 + c'_0)^4 |b|^2 C(\Omega)}{|a|^2 + \alpha C(\Omega)} \|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\|^2 \\
 &\quad + \frac{1}{2} \frac{|a|^2 + \alpha C(\Omega)}{|a|^2 C(\Omega)} \|\partial_t \theta^n\|^2 \\
 &\quad + \frac{1}{2} \frac{C(\Omega)}{|a|^2 + \alpha C(\Omega)} \|\partial_t P_h u^n - u^{n-\frac{1}{2}}\|^2 \\
 &\quad + \frac{1}{4} \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|^2.
 \end{aligned} \tag{6.9}$$

This yields

$$\frac{1}{2k} (|\theta^n|_1^2 - |\theta^{n-1}|_1^2) \leq C[(\sigma^n)^2 + (\delta^n)^2 + (\xi^n)^2]. \tag{6.10}$$

Using the estimates of σ^n , δ^n and ξ^n , we obtain

$$\begin{aligned}
 \frac{1}{2k} (|\theta^n|_1^2 - |\theta^{n-1}|_1^2) &\leq C \left[\|\theta^n\|^2 + \|\theta^{n-1}\|^2 + \|\rho^n\|^2 \right. \\
 &\quad \left. + \|\rho^{n-1}\|^2 + \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t(s)\|^2 ds \right. \\
 &\quad \left. + k^3 \int_{t_{n-1}}^{t_n} (\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|Du_{tt}(s)\|^2) ds \right].
 \end{aligned}$$

By (6.8) and (6.2.1), we get

$$|\theta^n|_1^2 - |\theta^{n-1}|_1^2 \leq Ck(h^r + k^2)^2 + CkR_n, \tag{6.11}$$

where

$$R_n = \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\rho_t(s)\|^2 ds + k^3 \int_{t_{n-1}}^{t_n} (\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|Du_{tt}(s)\|^2) ds.$$

Under the appropriate regularity assumptions for u , we have from (6.2.1) and (6.3)

$$|\theta^n|_1^2 \leq nkC(u)(h^r + k^2)^2 \leq TC(u)(h^r + k^2)^2,$$

or,

$$|\theta^n|_1 \leq C(u, T)(h^r + k^2), \quad 1 \leq n \leq N. \tag{6.12}$$

Applying Lemma 3 with $p = \infty$ and (6.12) and (6.8), we obtain

$$\|\theta^n\|_\infty \leq C(u, T)(h^r + k^2), \quad 1 \leq n \leq N, \tag{6.13}$$

and the result follows by (6.2.3). \square

7 The extrapolated Crank-Nicolson scheme

The above method has the disadvantage that a nonlinear system of algebraic equations has to be solved at each time step, as a result of the presence of $\varphi(U^n)$. For this reason we shall consider a linearized modification of the method in which the argument φ is obtained by extrapolation from U^{n-1} and U^{n-2} , or more precisely, with $\hat{U}^n = \frac{3}{2}U^{n-1} - \frac{1}{2}U^{n-2}$ for $n \geq 2$,

$$(\partial_t U^n, \chi) = -a(DU^{n-\frac{1}{2}}, D\chi) + b(\varphi(\hat{U}^n), \chi), \quad \forall \chi \in S_h. \quad (7.1)$$

This method will require a separate prescription for calculating U^1 , cf. e.g. [7] and [21, p 163]. We analyze a predictor corrector method for this purpose, using as a first approximation the value $U^{1,0}$ determined by the case $n = 1$ of (7.1) with \hat{U}^1 replaced by U^0 and then as the final approximation the result of the same equation with \hat{U}^1 replaced by $\frac{1}{2}(U^{1,0} + U^0)$, so that thus our starting procedure is defined by:

$$U^0 = u_{0h}, \quad (7.2)$$

followed by

$$\left(\frac{U^{1,0} - U^0}{k}, \chi \right) = -a \left(D \left(\frac{U^{1,0} + U^0}{2} \right), D\chi \right) + b(\varphi(U^0), \chi), \quad \forall \chi \in S_h, \quad (7.3)$$

and

$$(\partial_t U^1, \chi) = -a(DU^{\frac{1}{2}}, D\chi) + b \left(\varphi \left(\frac{U^{1,0} + U^0}{2} \right), \chi \right), \quad \forall \chi \in S_h. \quad (7.4)$$

Remark 1 For a smooth function $u(t)$, we have

$$\hat{u}^n = \frac{3}{2}u^{n-1} - \frac{1}{2}u^{n-2} = u^{n-\frac{1}{2}} + O(k^2) \quad \text{as } k \rightarrow 0. \quad (7.5)$$

Now we will prove that the proposed extrapolation will give the second order accuracy.

Theorem 6 Let U^n be the solution of (7.1) with U^0 and U^1 defined by (7.2), (7.3) and (7.4). Assume that (6.3) is valid and that the solution u of (1.2) is sufficiently smooth. Then, for h, k sufficiently small, we have the following estimate:

$$\|U^n - u(t^n)\|_\infty \leq C(h^r + k^2), \quad (7.6)$$

where $C = C(u, T, \Omega, \alpha, |a|, |b|, c_0, c'_0)$.

Proof First, we will prove that

$$\|\theta^n\| \leq C(h^r + k^2), \quad 1 \leq n \leq N. \quad (7.7)$$

For $n \geq 2$ and $\chi \in S_h$, we have

$$\begin{aligned} (\partial_t \theta^n, \chi) = & -a(D\theta^{n-\frac{1}{2}}, D\chi) + b(\varphi(\hat{U}^n) - \varphi(u^{n-\frac{1}{2}}), \chi) \\ & - (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \chi) - a\left(D\left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\right), D\chi\right). \end{aligned} \quad (7.8)$$

Setting $\chi = \theta^{n-\frac{1}{2}}$ in (7.8) and taking the real part, by (4.4), (5.4) and (1.2.4), we find

$$\frac{1}{2k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq C[\|\theta^{n-\frac{1}{2}}\|^2 + (\delta^n)^2 + (\xi^n)^2 + (\zeta^n)^2], \quad (7.9)$$

where

$$\zeta^n = \|\hat{U}^n - u^{n-\frac{1}{2}}\|.$$

Note that by (7.5)

$$\begin{aligned} \zeta^n & \leq \|\hat{\theta}^n\| + \|\hat{\rho}^n\| + \|\hat{u}^n - u^{n-\frac{1}{2}}\| \\ & \leq C(\|\theta^{n-1}\| + \|\theta^{n-2}\|) + C(u)(h^r + k^2). \end{aligned}$$

Using the estimates of δ^n , ξ^n , ζ^n , (6.2.1) and (7.9), we obtain

$$\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + Ck\|\theta^{n-2}\|^2 + C(u)k(h^r + k^2)^2.$$

Therefore,

$$\|\theta^n\|^2 + Ck\|\theta^{n-1}\|^2 \leq (1 + 2Ck)(\|\theta^{n-1}\|^2 + Ck\|\theta^{n-2}\|^2) + C(u)k(h^r + k^2)^2.$$

The above becomes, for $nk \leq T$ and $n \geq 2$,

$$\|\theta^n\|^2 \leq C[\|\theta^1\|^2 + k\|\theta^0\|^2 + (h^r + k^2)^2]. \quad (7.10)$$

Next we shall estimate $\|\theta^1\|$ from (7.3) and (7.4). We obtain instead of (6.7) from (7.3), with $\theta^{1,0} = U^{1,0} - P_h u^1$, $\theta^{0,0} = \theta^0$

$$\frac{1}{k}(\|\theta^{1,0}\|^2 - \|\theta^0\|^2) \leq C[\|U^0 - u^{\frac{1}{2}}\|^2 + (h^r + k^2)^2]. \quad (7.11)$$

It follows from (6.2.1) that

$$\begin{aligned} \|U^0 - u^{\frac{1}{2}}\| & \leq \|\theta^0\| + \|\rho^0\| + \|u^0 - u^{\frac{1}{2}}\| \\ & \leq \|\theta^0\| + C(h^r + k). \end{aligned}$$

Using (7.11) the above becomes

$$\frac{1}{k}(\|\theta^{1,0}\|^2 - \|\theta^0\|^2) \leq C[\|\theta^0\|^2 + h^{2r} + k^2].$$

Therefore,

$$\|\theta^{1,0}\|^2 \leq (1 + Ck)\|\theta^0\|^2 + Ck(h^{2r} + k^2) \leq C[\|\theta^0\|^2 + h^{2r} + k^3]. \quad (7.12)$$

In the same way as above we obtain instead of (6.7) from (7.4)

$$\frac{1}{k}(\|\theta^1\|^2 - \|\theta^0\|^2) \leq C \left[\left\| \frac{1}{2}(U^{1,0} + U^0) - u^{\frac{1}{2}} \right\|^2 + (h^r + k^2)^2 \right]. \quad (7.13)$$

Therefore, using (7.12)

$$\begin{aligned} \left\| \frac{1}{2}(U^{1,0} + U^0) - u^{\frac{1}{2}} \right\| &\leq \left\| \frac{1}{2}(\theta^{1,0} + \theta^0) \right\| + \|P_h u^{\frac{1}{2}} - u^{\frac{1}{2}}\| \\ &\leq \frac{1}{2}(\|\theta^{1,0}\| + \|\theta^0\|) + C(u)(h^r + k^2) \\ &\leq C\|\theta^0\| + C(u)(h^r + k^{\frac{3}{2}}). \end{aligned}$$

Hence from (7.13)

$$\|\theta^1\|^2 \leq (1 + Ck)\|\theta^0\|^2 + Ck(h^{2r} + k^3) \leq C[\|\theta^0\|^2 + (h^r + k^2)^2]. \quad (7.14)$$

It follows from (7.10) and (7.14), that from some constant $C = C(u, T)$ and $1 \leq n \leq N$

$$\|\theta^n\| \leq C[\|\theta^0\| + h^r + k^2], \quad (7.15)$$

from which the result (7.7) follows, in view of (6.3).

Second, we will prove that

$$|\theta^n|_1 \leq C(h^r + k^2), \quad 1 \leq n \leq N. \quad (7.16)$$

For $n \geq 2$, setting $\chi = \partial_t \theta^n$ in (7.8), multiplying by $\frac{1}{a}$ and taking the real part, we have

$$\begin{aligned} \frac{\alpha}{|a|^2} \|\partial_t \theta^n\|^2 &= -\frac{1}{2} \partial_t |\theta^n|_1^2 + \operatorname{Re} \left[\frac{b}{a} (\varphi(\hat{U}^{n-\frac{1}{2}}) - \varphi(u^{n-\frac{1}{2}}), \partial_t \theta^n) \right] \\ &\quad - \operatorname{Re} \left[\frac{1}{a} (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \partial_t \theta^n) \right] \\ &\quad - \operatorname{Re} \left[\left(D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right), D(\partial_t \theta^n) \right) \right]. \end{aligned}$$

It follows from Theorem 2 and (6.9) that

$$\begin{aligned} \frac{1}{2k} (|\theta^n|_1^2 - |\theta^{n-1}|_1^2) &\leq \frac{1}{2} \frac{(c_0 + c'_0)^4 |b|^2 C(\Omega)}{|a|^2 + \alpha C(\Omega)} \|\hat{U}^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2} \frac{C(\Omega)}{|a|^2 + \alpha C(\Omega)} \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{4} \left\| D \left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right) \right\|^2. \end{aligned}$$

Therefore,

$$\frac{1}{2k}(|\theta^n|_1^2 - |\theta^{n-1}|_1^2) \leq C[(\delta^n)^2 + (\xi^n)^2 + (\zeta^n)^2]. \quad (7.17)$$

Using the estimates of δ^n , ξ^n and ζ^n we obtain, under the appropriate regularity assumptions for u , and by (6.2.1), (6.3), (7.7)

$$\frac{1}{k}(|\theta^n|_1^2 - |\theta^{n-1}|_1^2) \leq C(u)(h^r + k^2)^2.$$

Consequently, for $n \geq 2$ and $nk \leq T$, we have

$$|\theta^n|_1 \leq C(u, T)[(h^r + k^2) + |\theta^1|_1]. \quad (7.18)$$

Next, we shall estimate $|\theta^1|_1$. Similarly to the analysis given above, we obtain instead of (6.10) from (7.3) and (7.4) respectively,

$$|\theta^1|_1 \leq C(u, T)(h^r + k^2). \quad (7.19)$$

Together (7.18) and (7.19) show the estimate (7.16). Applying Lemma 3 with $p = \infty$ and using (7.7) and (7.16), we find

$$\|\theta^n\|_\infty \leq C(u, T)(h^r + k^2), \quad 1 \leq n \leq N. \quad (7.20)$$

The proof of the Theorem is now concluded by (7.20) and (6.2.3). □

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